Solution. 1

(a) There are 4 independent choices, so 10^4.

(b) $10 \cdot 9 \cdot 8 \cdot 7$.

(c) Choose the remaining three: $9 \cdot 8 \cdot 7$.

(d) $\binom{10}{4}$.

(e) Pick two additional digits and count all permutations: $\binom{8}{2} \times 4!$.

These must be chosen in an unordered fashion, since we later count all $4!$ permutations of the unordered digits.

Solution. 2

It is easier to first count the number of forbidden shuffles. We have two different types of forbidden arrangements, see Fig. 2.

The number of decks with $A\heartsuit$ on top of $K\heartsuit$ is $51!$, since we can remove the $A\heartsuit$, shuffle the remaining 51 different cards, and then place the ace of hearts on top of the king of hearts. In the same manner, we have $51!$ forbidden decks involving $A\spadesuit$.

Finally, we need to count the number of elements in the intersection, i.e., decks where both of the forbidden configurations occur. We remove $A\heartsuit$ and $A\spadesuit$, shuffle the 50 cards, and insert the aces on the respective kings. This gives $50!$ shuffles. The number of forbidden configurations is therefore $51! + 51! - 50!$, and the total number of good decks is

$$52! - 2 \times 51! + 50!.$$

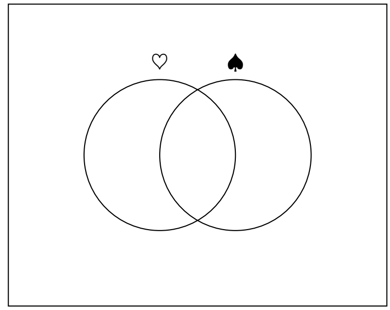

Figure 2: The total deck of cards, $52!$, and the two intersecting forbidden subsets.

Solution. 3

Since there are duplicates of E, S and I, there are $11!/2^3$ different words.

Solution. 4

It is given by the multinomial coefficient

$$\binom{7}{2,2,3} = \frac{7!}{2! \times 2! \times 3!} = 210.$$

Solution. 5

It is given by the multinomial coefficient

$$\binom{8}{2,2,2,2} = \frac{8!}{2^4}$$

It is the same as counting number of different words we can create from `AABBCCDD`. For example, the word `ADCBBCDA` assign dish A to person 1 and person 8.

Solution. 6

It is easier to consider the couples as labeled. We first pick 2 people to form couple A, then 2 other people to form couple B and so on. The number of ways to create *labeled* couples is

$$\binom{8}{2,2,2,2} = \frac{8!}{2^4}.$$

However, all permutations of the 4 labels produce the same set of couples, so we need to divide this by 4!. The answer is therefore $\frac{8!}{4! \times 2^4}$.

Solution. 7

First, we sample the 5 types. That leaves space for 8 more, which we can choose freely, with repetition. The dots-and-bars formula tells us that there are

$$\binom{8+5-1}{5-1} = \binom{12}{4}$$

ways to do this.

Solution. 9

Dots and bars give

$$\binom{r+n-1}{n-1}.$$

Solution. 10

Let $y_1 = x_1 - 2$, $y_2 = x_2 - 3$, $y_3 = x_3 - 10$ and $y_4 = x_4 + 3$.

We get a new equation where $y_i \geq 0$ and

$$(y_1 + 2) + (y_2 + 3) + (y_3 + 10) + (y_4 - 3) = 15, \qquad \Leftrightarrow$$
$$y_1 + y_2 + y_3 + y_4 = 3$$

Dots and bars gives $\binom{3+4-1}{4-1}$ integer solutions.

Solution. 11

We add one extra variable to turn the inequality to an equality:

$$x_1 + x_2 + x_3 + x_4 + s = 15, \qquad s, x_i \geq 0.$$

This gives $\binom{15+5-1}{5-1}$ integer solutions.

Solution. 13

There are $k!S(n,k)$ surjections — the quantity $k!$ is responsible for the labeling.

Solution. 16

This problem is not a clear-cut standard problem as in the introduction. However, the fact that each letter appears at least once imposes a lot of restriction, as we only need to decide which two additional letters to add to `abcd`.

This is a strong hint that we need to divide the problem into sub-cases. The fewer cases the better.

- We add two different letters. Thus, the letters appearing are one of `abbccd`, `abbcdd` or `abccdd`. To calculate the number of words we can make from `abbccd`, we use a multinomial coefficient, $\binom{6}{2,2,1,1}$.
- We add the same letter twice. This gives `abbbcd`, `abcccd` or `abcddd`, and each of these options give $\binom{6}{3,1,1,1}$ words.

Expanding the multinomials and putting it all together, there are in total

$$3\frac{6!}{(2!)^2}+3\frac{6!}{3!}$$

words satisfying the requirements.

Solution. 19

We notice that if the positions of the `b`'s and `d`'s are fixed, then the positions of the remaining letters is uniquely determined by the restriction. For example,

$$\texttt{b}\square\square\texttt{d}\square\texttt{b}\square\texttt{d} \quad\Longrightarrow\quad \texttt{baadcbcd}.$$

To create a valid word, it is enough to first choose 2 positions of the 8 available for the `b`'s and then 2 positions for the `d`'s. This can be done in $\binom{8}{2,2,4}=\frac{8!}{2!\times 2!\times 4!}$ ways.

Solution. 42

(a) There are $8\times 7\times\cdots\times 3$ ways.

(b) There are $(8\times 7\times 6)\times(5\times 4\times 3)$ ways.

(c) There are $\frac{1}{2}(8\times 7\times\cdots\times 3)$ ways, since the lines two pairs of lines (abc, def) and (def, abc) are considered equal configurations.

(d) There are $\binom{8}{6}$ ways.

(e) We first choose 6 people, and then choose 3 of these to be in group A: $\binom{8}{6}\binom{6}{3}$ ways.

(f) There are $\frac{1}{2}\binom{8}{6}\binom{6}{3}$ ways.

(g) There are $\binom{8}{6}\frac{1}{3!}\binom{6}{2,2,2}$ ways.

Solution. 43

(a) There is only one way.

(b) There is only one way.

(c) Bars and stars give $\binom{6+2}{2}$ ways.

(d) Bars and stars give $\binom{3+2}{2}$ ways.

(e) This is the same as counting integer partitions of 6 into 3 parts. We get three ways, $4+1+1$, $3+2+1$ and $2+2+2$.

(f) Same as previous question, but we can have as many piles as we like. We get 11 ways,

6	5+1	4+2	3+3
4+1+1	3+2+1	2+2+2	3+1+1+1
2+2+1+1	2+1+1+1+1	1+1+1+1+1+1.	

Solution. 44

(a) Each of the 6 spots has 8 options: 8^6.

(b) Also 8^6.

(c) We get $8^3 + \frac{1}{2} \times 8^3 \times (8^3 - 1)$ (the number of cases where groups are identical, plus number of cases where groups are different).

(d) Bars and stars give $\binom{6+8-1}{8-1}$ ways — the bars separate types.